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Petrov classification of class two space-times

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In this paper we have considered the Petrov classification of the space-times of class two. It is found that the metrics are either of Petrov type I or Petrov type II.

1. Introduction

The Riemann curvature tensor R_{hijk} plays the fundamental part in Einstein's theory of gravitation; the non-vanishing of at least one of its components at a point indicates the presence of the gravitational field at that point. The study of its algebraic structure has thrown a good deal of light on the properties of gravitational fields in general and the fields of gravitational radiation in particular. The essential physical idea behind the analysis of Riemann tensor is that this tensor represents the variation of gravitational fields from event to event in space-time through the equation of geodesic deviation. An algebraic classification of vacuum Riemann tensor was carried out by Petrov (1954). The relevance of Petrov's classification to the theory of gravitational radiation was suggested by Pirani (1957). He has shown that gravitational radiation in empty space-time is present if Riemann tensor belongs to either type II or III but not to type I in Petrov classification scheme. He has further extended this scheme of classification to the Weyl conformal curvature tensor C_{hijk} in order to include non-empty space-time in the discussion of gravitational radiation.

In this paper our object is to study the Petrov-Pirani classification of the conformal curvature tensors of space-times of class two in general. A space-time is said to be of class two if it can be embedded locally and isometrically in a pseudo-Euclidean space of six dimensions. The necessary and sufficient conditions for a space-time to be of embedding class two are that there exist two symmetric tensor a_{ij}, b_{ij} and a vector S_i satisfying following equations :

$$R_{hijk} = e_1(a_{hj}a_{ik}-a_{hk}a_{ij}+e_2(b_{hj}b_{ik}-b_{hk}b_{ij}) \quad \text{(Gauss equation)} \qquad \ldots \quad (1.1)$$

$$\left.\begin{aligned} a_{ij,\,k}-a_{ik;j} &= e_2(-s_k b_{ij}+s_j b_{ik}),\\ b_{ij;\,k}-b_{ik;\,j} &= -e_1(-s_k a_{ij}+s_j a_{ik}) \end{aligned}\right\} \quad \text{(Codazzi equations)}, \qquad \ldots \quad (1.2)$$

and

$$s_{i;\,j}-s_{j;i}+g^{kl}(a_{ki}b_{lj}-a_{kj}b_{li}) = 0 \qquad \text{(Ricci equation)}. \qquad \ldots \quad (1.3)$$

Here R_{hijk} and g_{kl} are curvature and metric tensor respectively of space-time ; e_1 and e_2 are real constants of modulus unity and semicolon in suffix denotes covariant differentiation.

If the elementary divisors of λ-matrices $[a_{ij}-g_{ij}]$ and $[b_{ij}-\lambda g_{ij}]$ are real, the tensor a_{ij}, b_{ij} and g_{ij} reduce simultaneously and locally to one of the following cases (Gupta 1971) :

Case 1 :

When the elementary divisors of $[a_{ij}-\lambda g_{ij}]$ and $[b_{ij}-\lambda g_{ij}]$ are simple and eigen vectors are collinear :

$$g_{ii} = e_i \quad (i = j), \qquad g_{ij} = 0 \quad (i \neq j),$$

and

$$e_i a_{ii} = \lambda_i^* \quad (i = j), \qquad a_{ij} = 0 \quad (i \neq j) \tag{1.4}$$

$$e_i b_{ii} = \lambda \quad (i = j), \qquad b_{ij} = 0 \quad (i \neq j),$$

λ_i^* and λ_i being eigen values of a_{ij} and b_{ij}; respectively. No summation for repeated suffiixes is implied.

Case 2 :

When the elementary divisors of $[b_{ij}-\lambda g_{ij}]$ are simple, whereas those of $[a_{ij}-\lambda g_{ij}]$ are multiple and the elemntary divisors of λ-matrix $[a_{ij}-\lambda g_{ij}]$ are of the form (1 1 2) we have,

$$g_{11} = -1, \quad g_{22} = -k_2 \quad g_{33} = -k_3, \quad g_{44} = 1,$$

$$b_{11} = -\lambda_1, \quad b_{22} = -k_2\lambda_2, \quad b_{33} = -k_3\lambda_3, \quad b_{44} = \lambda_4,$$

$$a_{11} = -k_1, \quad a_{22} = -k_2\lambda_2^*, \quad a_{33} = -k_3\lambda_3^*, \quad a_{44} = (2\lambda_1^* - k_1), \quad a_{14} = (k_1 - \lambda_1^*), \quad \dots \quad (1.5)$$

where k_1, k_2, k_3 are non-zero positive constants and λ_1^* is the multiple eigen value.

Case 3 :

When the elementary divisors of $[a_{ij}-\lambda g_{ij}]$ and $(b_{ij}-\lambda g_{ij}]$ are multiple, we have

$$g_{14} = 1, \quad g_{22} = -k_2, \quad g_{33} = -k_3,$$

$$b_{11} = k_1, \quad b_{14} = \lambda_1, \quad b_{22} = -k_2\lambda_2, \quad b_{33} = -k_3\lambda_3, \quad \dots \quad (1.6)$$

$$a_{11} = k_1, \quad a_{14} = \lambda_1^*, \quad a_{22} = -k_2\lambda_2^*, \quad a_{33} = -k_3\lambda_3^*,$$

where λ^*_1 and λ^*_1 are multiple eigen values.

The classification of the Weyl conformal tensors in space-times of class two reveals that in case 1, the space-time is of Petrov type I in general; in case 2, the conformal tensor is of type I but under certain restrictions it may be of Petrov type II; in case 3, the conformal tensor is of type I and type II in special cases. However, for the most general line elements of class two these three cases are not exhaustive. The more complicated cases will be dealt in a separate paper.

2. Petrov-Pirani Classification

Let $\lambda_{(a)}{}^m$ be a set of mutually orthogonal unit vectors associated with an event in space-time amongst which one is time-like and remaining three are space-like. The physical components of the conformal curvature tensor $C_{(abcd)}$ are given by

$$C_{(abcd)} = C_{hijk}\lambda_{(a)}{}^h\lambda_{(b)}{}^i\lambda_{(c)}{}^j\lambda_{(d)}{}^k. \quad \ldots \quad (2.1)$$

The Weyl conformal curvature tensor is defined as (Eisenhart 1949)

$$C_{hijk} = R_{hijk}+\tfrac{1}{2}(g_{hj}R_{ik}-g_{hk}R_{ij}+g_{ik}R_{hj}-g_{ij}R_{hk})+\frac{R}{6}(g_{hk}g_{ij}-g_{hj}g_{ik}) \quad \ldots \quad (2.2)$$

where

$$R_{ij} \equiv g^{mn}R_{mijn}, \quad R \equiv g^{ij}R_{ij}.$$

Case 1 :

From (1.1), (1.4) and (2.2) the non-vanishing components of conformal curvature tensor are

$$C_{1212} = \frac{1}{6}e_1[2(\lambda_1{}^*\lambda_2{}^*+\lambda_3{}^*\lambda_4{}^*)-(\lambda_1{}^*+\lambda_2{}^*)(\lambda_3{}^*+\lambda_4{}^*)]$$

$$+\frac{1}{6}e_2[2(\lambda_1\lambda_2+\lambda_3\lambda_4)-(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)],$$

$$C_{1313} = \frac{1}{6}e_1[2(\lambda_1{}^*\lambda_3{}^*+\lambda_2{}^*\lambda_4{}^*)-(\lambda_1{}^*+\lambda_3{}^*)(\lambda_2{}^*+\lambda_4{}^*)]$$

$$+\frac{1}{6}e_2[2(\lambda_1\lambda_3+\lambda_2\lambda_4)-(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)],$$

$$C_{2323} = \frac{1}{6}e_1[2(\lambda_2{}^*\lambda_3{}^*+\lambda_1{}^*\lambda_4{}^*)-(\lambda_2{}^*+\lambda_3{}^*)(\lambda_1{}^*+\lambda_4{}^*)]$$

$$+\frac{1}{6}e_2[2(\lambda_2\lambda_3+\lambda_1\lambda_4)-(\lambda_2+\lambda_3)(\lambda_1+\lambda_4)],$$

$$C_{1414} = -\frac{1}{6}e_1[2(\lambda_1{}^*\lambda_4{}^*+\lambda_2{}^*\lambda_3{}^*)-(\lambda_1{}^*+\lambda_4{}^*)(\lambda_2{}^*+\lambda_3{}^*)]$$

$$-\frac{1}{6}e_2[2(\lambda_1\lambda_4+\lambda_2\lambda_3)-(\lambda_1+\lambda_4)(\lambda_2+\lambda_3)],$$

$$C_{2424} = -\frac{1}{6}e_1[2(\lambda_2{}^*\lambda_4{}^*+\lambda_1{}^*\lambda_3{}^*)-(\lambda_2{}^*+\lambda_4{}^*)(\lambda_1{}^*+\lambda_3{}^*)]$$

$$-\frac{1}{6}e_2[2(\lambda_2\lambda_4+\lambda_1\lambda_3)-(\lambda_2+\lambda_4)(\lambda_1+\lambda_3)],$$

$$C_{3434} = -\frac{1}{6}e_1[2(\lambda_3{}^*\lambda_4{}^*+\lambda_1{}^*\lambda_2{}^*)-(\lambda_3{}^*+\lambda_4{}^*)(\lambda_1{}^*+\lambda_2{}^*)]$$

$$-\frac{1}{6}e_2[2(\lambda_3\lambda_4+\lambda_1\lambda_2)-(\lambda_3+\lambda_4)(\lambda_1+\lambda_2)]$$

Taking the tetrad in the direction of the axes the physical components of conformal curvature tensor are

$$C_{(2323)} = -C_{(1414)} = \alpha,$$

$$C_{(1313)} = -C_{(2424)} = \beta,$$

$$C_{(1212)} = -C_{(3434)} = \gamma,$$

where

$$\alpha = \frac{1}{6}e_1[2(\lambda_2{}^*\lambda_3{}^*+\lambda_1{}^*\lambda_4{}^*)-(\lambda_2{}^*+\lambda_3{}^*)(\lambda_1{}^*+\lambda_4{}^*)]$$

$$+\frac{1}{6}e_2[2(\lambda_2\lambda_3+\lambda_1\lambda_4)-(\lambda_2+\lambda_3)(\lambda_1+\lambda_4)],$$

$$\beta = \frac{1}{6}e_1[2(\lambda_1{}^*\lambda_3{}^*+\lambda_2{}^*\lambda_4{}^*)-(\lambda_1{}^*+\lambda_3{}^*)(\lambda_2{}^*+\lambda_4{}^*)]$$

$$+\frac{1}{6}e_2[2(\lambda_1\lambda_3+\lambda_2\lambda_4)-(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)],$$

$$\gamma = \frac{1}{6}e_1[2(\lambda_1{}^*\lambda_2{}^*+\lambda_3{}^*\lambda_4{}^*)-(\lambda_1{}^*+\lambda_2{}^*)(\lambda_3{}^*+\lambda_4{}^*)]$$

$$+\frac{1}{6}e_2[2(\lambda_1\lambda_2+\lambda_3\lambda_4)-(\lambda_1+\lambda_2)(\lambda_3+\lambda_4].$$

By relabelling the index pairs (a, b), (c, d) according to the following scheme in six dimensional formalism.

$$\begin{array}{lcccccc} (a\,b) : & 2\,3 & 3\,1 & 1\,2 & 1\,4 & 2\,4 & 3\,4 \\ A : & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad \ldots \quad (2.4)$$

the characteristic matrix can be written as

$$C_{[AB]}-\lambda\eta_{[AB]} = \text{diag}.\{(\alpha-\lambda), (\beta-\lambda), (\gamma-\lambda), -(\alpha-\lambda), -(\beta-\lambda), -(\gamma-\lambda)\} \quad \ldots \quad (2.5)$$

where η_{AB} is the Minkowskian matrix

$$\eta_{AB} = \text{diag}\,\{1, 1, 1, -1, -1, -1\}.$$

The Petrov type is given in the following table :

Table 1

Condition	Segre characteristic	Petrov type
(i) $\alpha \neq \beta \neq \gamma \neq 0$	[(1,1) (1,1) (1,1)]	I
(ii) Any two of α, β, γ are equal say $\beta = \gamma$	[(1,1,1,1) (1,1)]	I
(iii) $\alpha = \beta = \gamma$	[1 1 1 1 1 1]	I

It is obvious at once that in case when α, β and γ are zero, the space-time becomes conformally flat

Case 2 ·

From (1.1), (1.5) and (2.2) the non-vanishing components of conformal curvature tensor are

$$C_{1212} = k_2\Big[e_1\Big\{\tfrac{1}{2}k_1(\lambda_2^*-\lambda_3^*)-\frac{1}{6}\lambda_2^*(2\lambda_1^*+\lambda_3^*)-\frac{1}{6}\lambda_1^{2*}+\frac{2}{3}\lambda_1^*\lambda_3^*\Big\}$$

$$+e_2\Big\{\frac{1}{3}(\lambda_1\lambda_2+\lambda_3\lambda_4)-\frac{1}{6}(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)\Big\}\Big],$$

$$C_{1313} = k_3\Big[e_1\Big\{\tfrac{1}{2}k_1(\lambda_3^*-\lambda_2^*)-\frac{1}{6}\lambda_3^*(2\lambda_1^*+\lambda_2^*)-\frac{1}{6}\lambda_1^{*2}+\frac{2}{3}\lambda_1^*\lambda_2^*\Big\}$$

$$+e_2\Big\{\frac{1}{3}(\lambda_1\lambda_3+\lambda_2\lambda_4)-\frac{1}{6}(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)\Big\}\Big],$$

$$C_{1414} = e_1\Big[\frac{1}{3}(\lambda_1^*-\lambda_2^*)(\lambda_3^*-\lambda_1^*)\Big]+e_2\Big[-\frac{1}{3}(\lambda_1\lambda_4+\lambda_2\lambda_3)$$

$$+\frac{1}{6}(\lambda_1+\lambda_4)(\lambda_2+\lambda_3)\Big],$$

$$C_{1224} = \tfrac{1}{2}e_1k_2(k_1-\lambda_1^*)(\lambda_2^*-\lambda_3^*),$$

$$C_{1334} = -\tfrac{1}{2}e_1k_3(k_1-\lambda_1^*)(\lambda_2^*-\lambda_3^*),$$

$$C_{2323} = -k_2k_3\Big[\frac{1}{3}(\lambda_1{}^*-\lambda_2{}^*)(\lambda_3{}^*-\lambda_1{}^*)+e_2\Big\{-\frac{1}{3}(\lambda_1\lambda_4+\lambda_2\lambda_3)$$
$$+\frac{1}{6}(\lambda_1+\lambda_4)(\lambda_2+\lambda_3)\Big\}\Big],$$

$$C_{2424} = -k_2\Big[e_1\Big\{\tfrac{1}{2}k_1(\lambda_3{}^*-\lambda_2{}^*)-\frac{1}{6}\lambda_3{}^*(2\lambda_1{}^*+\lambda_2{}^*)-\frac{1}{6}\lambda_1{}^{2*}$$
$$+\frac{2}{3}\lambda_1{}^*\lambda_2{}^*\Big\}+e_2\Big\{\frac{1}{3}(\lambda_1\lambda_3+\lambda_2\lambda_4)-\frac{1}{6}(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)\Big\}\Big],$$

$$C_{3434} = -k_3\Big[e_1\Big\{\tfrac{1}{2}k_1(\lambda_2{}^*-\lambda_3{}^*)-\frac{1}{6}\lambda_2{}^*(2\lambda_1{}^*+\lambda_3{}^*)-\frac{1}{6}\lambda_1{}^{*2}$$
$$+\frac{2}{3}\lambda_1{}^*\lambda_3{}^*\Big\}+e_2\Big\{\frac{1}{3}(\lambda_1\lambda_2+\lambda_3\lambda_4)-\frac{1}{6}(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)\Big\}\Big], \quad \ldots \quad (2.6)$$

Choosing the tetrad

$$\lambda_{(a)}{}^k = \text{diag.}\Big\{1, \frac{1}{\sqrt{k_2}}, \frac{1}{\sqrt{k_3}}, -1\Big\}$$

we obtain from (2.6) the physical components of conformal curvature tensor as

$$\begin{aligned} C_{(2323)} &= -C_{(1414)} = a, \\ C_{(1313)} &= -C_{(2424)} = b, \\ C_{(1212)} &= -C_{(3434)} = c, \\ C_{(1224)} &= -C_{(1334)} = d, \end{aligned}$$

where

$$a = -\Big[\frac{1}{3}(\lambda_1{}^*-\lambda_2{}^*)(\lambda_3{}^*-\lambda_1{}^*)+e_2\Big\{-\frac{1}{3}(\lambda_1\lambda_4+\lambda_2\lambda_3)$$
$$+\frac{1}{6}(\lambda_1+\lambda_4)(\lambda_2+\lambda_3)\Big\}\Big],$$

$$b = \Big[e_1\Big\{\tfrac{1}{2}k_1(\lambda_3{}^*-\lambda_2{}^*)-\frac{1}{6}\lambda_2{}^*(2\lambda_1{}^*+\lambda_2{}^*)-\frac{1}{6}\lambda_1{}^{2*}+\frac{2}{3}\lambda_1{}^*\lambda_2{}^*\Big\}$$
$$+e_2\Big\{\frac{1}{3}(\lambda_1\lambda_3+\lambda_2\lambda_4)-\frac{1}{6}(\lambda_1+\lambda_3)(\lambda_2+\lambda_4)\Big\}\Big],$$

$$c = \Big[e_1\Big\{\tfrac{1}{2}k_1(\lambda_2{}^*-\lambda_3{}^*)-\frac{1}{6}\lambda_2{}^*(2\lambda_1{}^*+\lambda_3{}^*)-\frac{1}{6}\lambda_1{}^{*2}+\frac{2}{3}\lambda_1{}^*\lambda_3{}^*\Big\}$$
$$+e_2\Big\{\frac{1}{3}(\lambda_1\lambda_2+\lambda_3\lambda_4)-\frac{1}{6}(\lambda_1+\lambda_2)(\lambda_3+\lambda_4)\Big\}\Big],$$

$$d = \tfrac{1}{2}e_1(k_1-\lambda_1{}^*)(\lambda_2{}^*-\lambda_3{}^*). \quad \ldots \quad (2.7)$$

One can easily verify that

$$a = -(b+c).$$

By relabelling the index pairs (a, b), (c, d) according to scheme (2.3) we obtain from (2.7) the λ-matrix

$$C_{[AB]}-\lambda\eta_{[AB]} = \left| \begin{array}{cccccc} a-\lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & b-\lambda & 0 & 0 & 0 & d \\ 0 & 0 & c-\lambda & 0 & d & 0 \\ 0 & 0 & 0 & -(a-\lambda) & 0 & 0 \\ 0 & 0 & d & 0 & -(b-\lambda) & 0 \\ 0 & d & 0 & 0 & 0 & -(c- \end{array} \right. \quad (2.8)$$

It may be noted that if $d = 0$, the λ-matrix (2.8) is identical with the λ-matrix (2.5) which has already been classified.

With the help of elementary λ-transformations the λ-matrix (2.8) can be obtained in the form

$$\text{diag.}\{d, d, (a-\lambda), -(a-\lambda), \delta, \delta\},$$

where

$$\delta = d^2+(\lambda-b)(\lambda-c).$$

The various Petrov types are given in the following table :

Table 2

	Conditions	Segre characteristic	Petrov type
(1)	$a, b, c, d \neq 0$ $2a+bc+d^2 \neq 0$	[(1,1) (1,1) (1,1)]	I
(2)	$a, b, c, d \neq 0$ $2a+bc+d^2 \neq 0$ $(b-c)^2-4d^2 = 0$	[(1,1) (2,2)]	II
(3)	$a \neq b \neq c \neq 0, \quad d = 0$	[(1,1) (1,1) (1,1)]	I
(4)	Any two of a, b, c are equal say $b = c, \quad d = 0$	[(1,1,1,1) (1,1)]	I
(5)	$a = 0, b, c, d \neq 0$	[(1,1) (1,1) (1,1)]	I
(6)	$a = 0, \quad b, c, d \neq 0,$ $b^2-d^2 = 0$	[(1,1) (2,2)]	II
(7)	$b = 0, a, c, d \neq 0$	[(1,1) (1,1) (1,1)]	I
(8)	$b = 0, a, c, d \neq 0, a^2-4d = 0$	[(1,1) (2,2)]	II
(9)	$a = 0, \quad d = 0$	[(1,1) (1,1) (1,1)]	I
(10)	$a = b = 0, d \neq 0$	[(1,1) (1,1) (1,1)]	I

It is clear that when a, b, c, d are all zero, the space-time is conformally

Case 3 :

In this case the non-vanishing components of conformal curvature tensor are

$$
\begin{aligned}
C_{1212} &= -\tfrac{1}{2}k_1k_2[e_1(\lambda_2{}^*-\lambda_3{}^*)+e_2(\lambda_2-\lambda_3)],\\
C_{1313} &= \tfrac{1}{2}k_1k_3[e_1(\lambda_2{}^*-\lambda_3{}^*)+e_2(\lambda_2-\lambda_3)],\\
C_{2323} &= \tfrac{1}{3}k_2k_3[e_2(\lambda_2{}^*\lambda_3{}^*-\lambda_1{}^*\lambda_2{}^*-\lambda_1{}^*\lambda_3{}^*+\lambda_1{}^{*2})\\
&\qquad\qquad +e_2(\lambda_2\lambda_3-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_1{}^2)],\\
C_{1414} &= -\tfrac{1}{3}[e_1(\lambda_2{}^*\lambda_3{}^*-\lambda_1{}^*\lambda_2{}^*-\lambda_1{}^*\lambda_3{}^*+\lambda_1{}^{*2})\\
&\qquad +e_2(\lambda_2\lambda_3-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_1{}^2)],\\
C_{1224} &= -\tfrac{1}{6}k_2[e_1(\lambda_2{}^*\lambda_3{}^*-\lambda_1{}^*\lambda_2{}^*-\lambda_1{}^*\lambda_3{}^*+\lambda_1{}^{*2})\\
&\qquad +e_2(\lambda_2\lambda_3-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_1{}^2)],\\
C_{1334} &= -\tfrac{1}{6}k_3[e_1(\lambda_2{}^*\lambda_3{}^*-\lambda_1{}^*\lambda_2{}^*-\lambda_1{}^*\lambda_3{}^*+\lambda_1{}^{*2})\\
&\qquad +e_2(\lambda_2\lambda_3-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_1{}^2)],
\end{aligned}
$$

Choosing the tetrad $\lambda_{(a)}{}^m$:

$$
\left(\frac{1}{\sqrt{2}},0,0,-\frac{1}{\sqrt{2}}\right),\ \left(0,-\frac{1}{\sqrt{k_2}},0,0\right),\ \left(0,0,-\frac{1}{\sqrt{k_3}},0\right)\left(\frac{1}{\sqrt{2}}\,0,\ 0,\frac{1}{\sqrt{2}}\right)
$$

we obtain the physical components of conformal tensor

$$
\begin{aligned}
C_{(2323)} &= -C_{(1414)} = \mu,\\
C_{(1313)} &= -C_{(2424)} = \rho,\\
C_{(1212)} &= -C_{(3434)} = \sigma,\\
C_{(1224)} &= -C_{(1334)} = \nu,
\end{aligned}
$$

where

$$
\begin{aligned}
\mu &= \tfrac{1}{3}[e_1(\lambda_2{}^*\lambda_3{}^*-\lambda_1{}^*\lambda_2{}^*-\lambda_1{}^*\lambda_3{}^*+\lambda_1{}^{*2})\\
&\qquad +e_2(\lambda_2\lambda_3-\lambda_1\lambda_2-\lambda_1\lambda_3+\lambda_1{}^2)],\\
\rho &= \tfrac{1}{12}[e_1(3k_1\lambda_2{}^*-3k_1\lambda_3{}^*-2\lambda_2{}^*\lambda_3{}^*+2\lambda_1{}^*\lambda_2{}^*-2\lambda_1{}^*\lambda_3{}^*\\
&\qquad -2\lambda_1{}^{*2})+e_2[3k_1\lambda_2-3k_1\lambda_3-2\lambda_2\lambda_3+2\lambda_1\lambda_2+2\lambda_1\lambda_3-2\lambda_1{}^2)],\\
\sigma &= -\tfrac{1}{12}[e_1(3k_1\lambda_2{}^*-3k_1\lambda_3{}^*+2\lambda_2{}^*\lambda_3{}^*-2\lambda_1{}^*\lambda_2{}^*-2\lambda_1{}^*\lambda_3{}^*+2\lambda_1{}^{*2})\\
&\qquad +e_2(3k_1\lambda_2-3k_1\lambda_3+2\lambda_2\lambda_3-2\lambda_1\lambda_2-2\lambda_1\lambda_3+2\lambda_1{}^2)],\\
\nu &= \tfrac{1}{4}k_1[e_1(\lambda_2{}^*-\lambda_3{}^*)+e_2(\lambda_2-\lambda_3)].
\end{aligned}
$$

By relabelling the index pairs (a, b), $(c\, d)$ according to the scheme (2.3) we obtain from (2.10) the characteristic matrix :

$$C_{[AB]}-\lambda\eta_{[AB]} = \begin{array}{cccccc} \mu-\lambda & 0 & 0 & 0 & 0 & \\ 0 & \rho-\lambda & 0 & 0 & 0 & \\ 0 & 0 & \sigma-\lambda & 0 & \nu & 0 \\ 0 & 0 & 0 & -(\mu-\lambda) & 0 & 0 \\ 0 & 0 & \nu & 0 & -(\rho-\lambda) & 0 \\ 0 & & 0 & 0 & & (\sigma-\lambda) \end{array} \qquad \ldots (2.11)$$

. is easy to see that

$$\mu = -(\rho+\sigma).$$

The λ-matrix (2.11) is identical with the matrix (2.8). Thus in case 3, we get similar type of classification as obtained in the case 2

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